Connectivity of Large Wireless Networks under A Generic Connection Model

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***Abstract*—This paper studies networks where all nodes are distributed on a unit square $A \triangleq [-\frac{1}{2}, \frac{1}{2}]^2$ following a Poisson distribution with known density ρ and a pair of nodes separated by an Euclidean distance x are directly connected with probability $g_{r_\rho}(x) \triangleq g(x/r_\rho)$, independent of the event that any other pair of nodes are directly connected. Here $g : [0, \infty) \to [0, 1]$ satisfies the conditions of rotational invariance, non-increasing monotonicity, integral boundedness and $g(x) = o(1/(x^2 \log^2 x))$; further, $r_\rho = \sqrt{(\log \rho + b)/(C\rho)}$ where $C = \int_{\Re^2} g(\|\boldsymbol{x}\|)d\boldsymbol{x}$ and b is a constant. Denote the above network by $\mathcal{G}(\mathcal{X}_\rho, g_{r_\rho}, A)$. We show that as $\rho \to \infty$, a) the distribution of the number of isolated nodes in $\mathcal{G}(\mathcal{X}_\rho, g_{r_\rho}, A)$ converges to a Poisson distribution with mean e^{-b}; b) asymptotically almost surely (*a.a.s.*) there is no component in $\mathcal{G}(\mathcal{X}_\rho, g_{r_\rho}, A)$ of fixed and finite order $k > 1$; c) *a.a.s.* the number of components with an unbounded order is one. Therefore as $\rho \to \infty$, the network *a.a.s.* contains a unique unbounded component and isolated nodes only; a sufficient and necessary condition for $\mathcal{G}(\mathcal{X}_\rho, g_{r_\rho}, A)$ to be *a.a.s.* connected is that there is no isolated node in the network, which occurs when $b \to \infty$ as $\rho \to \infty$. These results expand recent results obtained for connectivity of random geometric graphs from the unit disk model and the fewer results from the log-normal model to the more generic and more practical random connection model.**

***Index Terms*—Connectivity, random geometric graph, random connection model**

I. Introduction

Connectivity is one of the most fundamental properties of wireless multi-hop networks [2]–[4]. A network is said to be *connected* if there is a path between any pair of nodes.

Extensive research has been done on connectivity problems using the well-known random geometric graph and the unit disk connection model, which is usually obtained by randomly and uniformly distributing n vertices in a given area and connecting any two vertices iff (if and only if) their Euclidean distance is smaller than or equal to a given threshold $r(n)$ [3], [5]. Significant outcomes have been obtained [2], [3], [6]. Particularly, Penrose [7], [8] and Gupta and Kumar [2] proved using different techniques that if the transmission range is set to $r(n) = \sqrt{(\log n + c(n))/(\pi n)}$, a random network formed by uniformly placing n nodes on a unit-area disk in $\Re^2$ is asymptotically almost surely (*a.a.s.*) connected as $n \to \infty$ iff $c(n) \to \infty$. [An event ξ is said to occur *almost surely* if its probability equals to one; an event ξ_n depending on n is said to occur *a.a.s.* if its probability tends to one as $n \to \infty$]. Specifically, Penrose's result is based on the fact that in the above random network as $n \to \infty$ the longest edge of the minimum spanning tree converges in probability to the minimum transmission range required for the above network to have no isolated nodes [3], [7], [8]. Gupta and Kumar's result is based on a key finding in continuum percolation theory [9, Chapter 6]: consider an *infinite* network with nodes distributed on $\Re^2$ following a Poisson distribution with density ρ; and suppose that a pair of nodes separated by an Euclidean distance x are directly connected with probability $g(x)$, independent of the event that another distinct pair of nodes are directly connected. Here, $g : \Re^+ \to [0, 1]$ satisfies the conditions of rotational invariance, non-increasing monotonicity and integral boundedness [9, pp. 151-152]. Denote the above network by $\mathcal{G}(\mathcal{X}_\rho, g, \Re^2)$. As $\rho \to \infty$, *a.a.s.* $\mathcal{G}(\mathcal{X}_\rho, g, \Re^2)$ has only a unique infinite component and isolated nodes. The work of Gupta and Kumar is however incomplete to the extent that the above result obtained in continuum percolation theory for an infinite network cannot, counter to intuition, be directly applied to a finite (or asymptotically infinite) network on a finite (or asymptotically infinite) area in $\Re^2$ [10].

In addition to the above work based on the unit disk connection model, there is also limited work [11], [12] dealing with the necessary condition for a random network to be connected under the log-normal shadowing connection model. Under the log-normal shadowing connection model, two nodes are directly connected if the received power at one node from the other node, whose attenuation follows the log-normal model, is greater than a given threshold. The results in [11], [12] however rely on the assumption that the node isolation events are independent. This assumption has only been justified using simulations.

Some work also exists on the analysis of the asymptotic distribution of the number of isolated nodes [3], [13]–[15] under the assumption of a unit disk model. In [13], Yi et al. considered a total of n nodes distributed independently and uniformly on a unit-area disk and each node may be active independently with some probability p. A node is considered to be isolated if it is not directly connected to any of the active nodes. Using some complicated geometric analysis, they showed that if all nodes have a maximum transmission

Some results in Section III of this paper appeared in INFOCOM 2011 [1]. Substantial improvements have been made on the theoretical analysis in [1].

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range $r(n) = \sqrt{(\log n + \xi)/(\pi p n)}$ for some constant ξ, the total number of isolated nodes is asymptotically Poissonly distributed with mean $e^{-\xi}$. In [14], [15], Franceschetti et al. derived essentially the same result using the Chen-Stein technique. A similar result can also be found in the earlier work of Penrose [3] in a continuum percolation setting.

In this paper, we consider a network where all nodes are distributed on a unit square $A \triangleq [-\frac{1}{2}, \frac{1}{2}]^2$ following a Poisson distribution with known density ρ and a pair of nodes are directly connected following a *generic random connection model* g_{r_ρ}, to be rigorously defined in Section II. Denote the above network by $\mathcal{G}\left(\mathcal{X}_\rho, g_{r_\rho}, A\right)$, where $\mathcal{X}_\rho$ denotes the set of nodes in the network. We give the sufficient and necessary condition for $\mathcal{G}\left(\mathcal{X}_\rho, g_{r_\rho}, A\right)$ to be *a.a.s.* connected as $\rho \to \infty$. The results in this paper expand the above results on network connectivity to a more generic random connection model, with the unit disk model and the log-normal model being two special cases, thus providing an important link that allows the expansion of other associated results on connectivity to the random connection model.

The main contributions of this paper are:

1) Using the Chen-Stein technique [16], [17], we show that the distribution of the number of isolated nodes in $\mathcal{G}\left(\mathcal{X}_\rho, g_{r_\rho}, A\right)$ asymptotically converges to a Poisson distribution as $\rho \to \infty$. This result readily leads to a necessary condition for $\mathcal{G}\left(\mathcal{X}_\rho, g_{r_\rho}, A\right)$ to be *a.a.s.* connected as $\rho \to \infty$;
2) We show that as $\rho \to \infty$, the number of components in $\mathcal{G}\left(\mathcal{X}_\rho, g_{r_\rho}, A\right)$ of unbounded order converges to one. This result, together with the result in [10] that the number of components of finite order $k > 1$ in $\mathcal{G}\left(\mathcal{X}_\rho, g_{r_\rho}, A\right)$ asymptotically vanishes as $\rho \to \infty$, allows us to conclude that as $\rho \to \infty$, *a.a.s.* there are only a unique unbounded component and isolated nodes in $\mathcal{G}\left(\mathcal{X}_\rho, g_{r_\rho}, A\right)$.
3) The above results allow us to establish that the sufficient and necessary condition for $\mathcal{G}\left(\mathcal{X}_\rho, g_{r_\rho}, A\right)$ to be *a.a.s.* connected is that there is no isolated node in the network. On that basis, we obtain the asymptotic probability that $\mathcal{G}\left(\mathcal{X}_\rho, g_{r_\rho}, A\right)$ forms a connected network as $\rho \to \infty$ and the sufficient and necessary condition for $\mathcal{G}\left(\mathcal{X}_\rho, g_{r_\rho}, A\right)$ to be *a.a.s.* connected.

The rest of this paper is organized as follows: Section II introduces the network model and problem setting; Section III establishes a necessary condition for $\mathcal{G}\left(\mathcal{X}_\rho, g_{r_\rho}, A\right)$ to be *a.a.s.* connected; Section IV first establishes a sufficient condition for $\mathcal{G}\left(\mathcal{X}_\rho, g_{r_\rho}, A\right)$ to be *a.a.s.* connected and on that basis, together with the results in Section III, then establishes the sufficient and necessary condition for $\mathcal{G}\left(\mathcal{X}_\rho, g_{r_\rho}, A\right)$ to be *a.a.s.* connected; finally Section V concludes the paper.

II. Network Model and Problem Setting

We consider a network where all nodes are distributed on a unit square $A \triangleq [-\frac{1}{2}, \frac{1}{2}]^2$ following a Poisson distribution with known density ρ and a pair of nodes are directly connected following a *random connection model*, viz. a pair of nodes separated by an Euclidean distance x are directly connected with probability $g_{r_\rho}(x) \triangleq g(x/r_\rho)$, where $g : [0, \infty) \to [0, 1]$, independent of the event that another pair of nodes are directly connected. Here

$$r_\rho = \sqrt{(\log \rho + b)/(C\rho)} \tag{1}$$

and b is a constant. The reason for choosing this particular form of r_ρ is that the analysis becomes nontrivial when b is a constant. Other forms of r_ρ can be accommodated by dropping the assumption that b is constant, i.e. b becomes a function of ρ, and allowing $b \to \infty$ or $b \to -\infty$ as $\rho \to \infty$. The results are rapidly attainable, and we discuss these situations separately in Sections III and IV.

The function g is usually required to satisfy the following properties of monotonicity, integral boundedness and rotational invariance [9], [15, Chapter 6][1]:

$$g\left(x\right) \leq g\left(y\right) \qquad whenever \;\; x \geq y \tag{2}$$

$$0 < C \triangleq \int_{\Re^2} g(\|\boldsymbol{x}\|) d\boldsymbol{x} < \infty \tag{3}$$

where $\|\|$ represents the Euclidean norm. We refer readers to [9], [15, Chapter 6] for detailed discussions on the random connection model.

Equations (2) and (3) allow us to conclude that [10, Equation (3)][2]

$$g\left(x\right) = o_x(1/x^2) \tag{4}$$

However, we require g to satisfy the more restrictive requirement that

$$g\left(x\right) = o_x(1/(x^2 \log^2 x)) \tag{5}$$

The condition (5) is only slightly more restrictive than (4) in that for an arbitrarily small positive constant ε, $1/x^{2+\varepsilon} = o_x(1/(x^2 \log^2 x))$. The more restrictive requirement is needed to ensure that the impact of the *truncation effect* on connectivity is asymptotically vanishingly small as $\rho \to \infty$ [10].

For convenience we also assume that g has infinite support when necessary. Our results however apply to the situation when g has bounded support, which forms a special case and actually makes the analysis easier.

Denote the above network by $\mathcal{G}\left(\mathcal{X}_\rho, g_{r_\rho}, A\right)$. It is obvious that under a *unit disk model* where $g(x) = 1$ for $x \leq 1$ and $g(x) = 0$ for $x > 1$, r_ρ corresponds to the critical transmission range for connectivity [2]. Thus the above model incorporates the unit disk model as a special case. A similar conclusion can also be drawn for the log-normal connection model.

[1]Throughout this paper, we use the non-bold symbol, e.g. x, to denote a scalar and the bold symbol, e.g. $\boldsymbol{x}$, to denote a vector.

[2]The following notations and definitions are used throughout the paper:

- $f\left(z\right) = o_z(h\left(z\right))$ iff $\lim_{z\to\infty} \frac{f(z)}{h(z)} = 0$;
- $f\left(z\right) = \omega_z(h\left(z\right))$ iff $h\left(z\right) = o_z\left(f\left(z\right)\right)$;
- $f\left(z\right) = \Theta_z(h\left(z\right))$ iff there exist a sufficiently large z_0 and two positive constants c_1 and c_2 such that for any $z > z_0$, $c_1 h\left(z\right) \geq f\left(z\right) \geq c_2 h\left(z\right)$;
- $f\left(z\right) \sim_z h\left(z\right)$ iff $\lim_{z\to\infty} \frac{f(z)}{h(z)} = 1$;

III. Necessary Condition for *a.a.s.* Connected Network

In this section, as an intermediate step to obtaining the main result, we first and temporarily consider a network with the same node distribution and connection model as $\mathcal{G}\left(\mathcal{X}_\rho, g_{r_\rho}, A\right)$ however with nodes deployed on a unit torus $A^T \triangleq [-\frac{1}{2}, \frac{1}{2}]^2$. Denote the network on the torus by $\mathcal{G}^T\left(\mathcal{X}_\rho, g_{r_\rho}, A\right)$. We show that as $\rho \to \infty$, the distribution of the number of isolated nodes in $\mathcal{G}^T\left(\mathcal{X}_\rho, g_{r_\rho}, A\right)$, denoted by W^T, asymptotically converges to a Poisson distribution with mean e^{-b}. We then extend the above result to $\mathcal{G}\left(\mathcal{X}_\rho, g_{r_\rho}, A\right)$. On that basis, we obtain a necessary condition for $\mathcal{G}\left(\mathcal{X}_\rho, g_{r_\rho}, A\right)$ to be *a.a.s.* connected as $\rho \to \infty$.

A. Distribution of the number of isolated nodes on a torus

In this subsection, we analyze the distribution of the number of isolated nodes in $\mathcal{G}^T\left(\mathcal{X}_\rho, g_{r_\rho}, A\right)$.

The use of a toroidal rather than planar region as a tool in analyzing network properties is well known [3]. The unit torus $A^T = [-\frac{1}{2}, \frac{1}{2}]^2$ that is commonly used in random geometric graph theory is essentially the same as a unit square $A = [-\frac{1}{2}, \frac{1}{2}]^2$ except that the distance between two points on a torus is defined by their *toroidal distance*, instead of Euclidean distance. Thus a pair of nodes in $\mathcal{G}^T\left(\mathcal{X}_\rho, g_{r_\rho}, A\right)$, located at $\boldsymbol{x}_1$ and $\boldsymbol{x}_2$ respectively, are directly connected with probability $g_{r_\rho}(\|\boldsymbol{x}_1 - \boldsymbol{x}_2\|^T)$ where $\|\boldsymbol{x}_1 - \boldsymbol{x}_2\|^T$ denotes the *toroidal distance* between the two nodes. For a unit torus $A^T = [-\frac{1}{2}, \frac{1}{2}]^2$, the toroidal distance is given by [3, p. 13]:

$$\|\boldsymbol{x}_1 - \boldsymbol{x}_2\|^T \triangleq \min\{\|\boldsymbol{x}_1 + \boldsymbol{z} - \boldsymbol{x}_2\| : \boldsymbol{z} \in \mathbb{Z}^2\} \tag{6}$$

In this section, whenever the difference between a torus and a square affects the parameter being discussed, we use superscript ^{T} to mark the parameter in a torus while the unmarked parameter is associated with a square.

We note the following relation between toroidal distance and Euclidean distance on a square area centered at the origin:

$$\|\boldsymbol{x}_1 - \boldsymbol{x}_2\|^T \le \|\boldsymbol{x}_1 - \boldsymbol{x}_2\| \quad \text{and} \quad \|\boldsymbol{x}\|^T = \|\boldsymbol{x}\| \tag{7}$$

which will be used in the later analysis.

The main result of this subsection is given in Theorem 1.

Theorem 1. *The distribution of the number of isolated nodes in $\mathcal{G}^T\left(\mathcal{X}_\rho, g_{r_\rho}, A\right)$ converges to a Poisson distribution with mean e^{-b} as $\rho \to \infty$.*

Proof: See Appendix I. ■

B. Distribution of the number of isolated nodes on a square

We now consider the asymptotic distribution of the number of isolated nodes in $\mathcal{G}\left(\mathcal{X}_\rho, g_{r_\rho}, A\right)$.

Let W be the number of isolated nodes in $\mathcal{G}\left(\mathcal{X}_\rho, g_{r_\rho}, A\right)$ and W^E be the number of isolated nodes in $\mathcal{G}\left(\mathcal{X}_\rho, g_{r_\rho}, A\right)$ due to the boundary effect. Using the coupling technique, it can be readily shown that $W = W^E + W^T$ [10]. Using the above equation, Theorem 1, Lemma 2 in [10][3], which showed that $\lim_{\rho\to\infty} \Pr(W^E = 0) = 1$, and Slutsky's theorem [18], the following result on the asymptotic distribution of W can be readily obtained.

Theorem 2. *The distribution of the number of isolated nodes in $\mathcal{G}\left(\mathcal{X}_\rho, g_{r_\rho}, A\right)$ converges to a Poisson distribution with mean e^{-b} as $\rho \to \infty$.*

Corollary 3 follows immediately from Theorem 2.

Corollary 3. *As $\rho \to \infty$, the probability that there is no isolated node in $\mathcal{G}\left(\mathcal{X}_\rho, g_{r_\rho}, A\right)$ converges to $e^{-e^{-b}}$.*

Now we relax requirement that b is a constant to obtain a necessary condition for $\mathcal{G}\left(\mathcal{X}_\rho, g_{r_\rho}, A\right)$ to be *a.a.s.* connected. Specifically, consider the situation when $b \to -\infty$ or $b \to \infty$ as $\rho \to \infty$. Note that the property that the network $\mathcal{G}\left(\mathcal{X}_\rho, g_{r_\rho}, A\right)$ has no isolated node is an *increasing* property (For an arbitrary network, a particular property is termed *increasing* if the property is preserved when more connections (edges) are added into the network.). Using a coupling technique similar to that used in [15, Chapter 2] and with a few simple steps (omitted), the following theorem and corollary can be obtained, which form a major contribution of this paper:

Theorem 4. *In $\mathcal{G}\left(\mathcal{X}_\rho, g_{r_\rho}, A\right)$, if $b \to \infty$ as $\rho \to \infty$, a.a.s. there is no isolated node in the network; if $b \to -\infty$ as $\rho \to \infty$, a.a.s. the network has at least one isolated node.*

Corollary 5. *$b \to \infty$ is a necessary condition for $\mathcal{G}\left(\mathcal{X}_\rho, g_{r_\rho}, A\right)$ to be a.a.s. connected as $\rho \to \infty$.*

IV. Sufficient Condition for *a.a.s.* Connected Network

In this section, we continue to investigate the sufficient condition for $\mathcal{G}\left(\mathcal{X}_\rho, g_{r_\rho}, A\right)$ to be *a.a.s.* connected. In [10] we showed that vanishing of components of finite order $k > 1$ in $\mathcal{G}\left(\mathcal{X}_\rho, g, \Re^2\right)$ as $\rho \to \infty$ (as shown in [9, Theorems 6.3]) does not *necessarily* carry the conclusion that components of finite order $k > 1$ in $\mathcal{G}\left(\mathcal{X}_\rho, g_{r_\rho}, A\right)$ also vanish as $\rho \to \infty$, contrary perhaps to intuition. Then, we presented a result for the vanishing of components of finite order $k > 1$ in $\mathcal{G}\left(\mathcal{X}_\rho, g_{r_\rho}, A\right)$ as $\rho \to \infty$ to fill this theoretical gap [10, Theorem 4]. On the basis of the above results, we shall further demonstrate in this section that *a.a.s.* the number of unbounded components in $\mathcal{G}\left(\mathcal{X}_\rho, g_{r_\rho}, A\right)$ is one as $\rho \to \infty$. A sufficient condition for $\mathcal{G}\left(\mathcal{X}_\rho, g_{r_\rho}, A\right)$ to be *a.a.s.* connected readily follows.

In [9, Theorem 6.3], it was shown that there can be at most one unbounded component in $\mathcal{G}\left(\mathcal{X}_\rho, g, \Re^2\right)$. However

[3]Let $\mathcal{G}\left(\mathcal{X}_\lambda, g, A_{\frac{1}{r_\rho}}\right)$ be a network with nodes Poissonly distributed on a square $A_{\frac{1}{r_\rho}} = [-\frac{1}{2r_\rho}, \frac{1}{2r_\rho}]^2$ with density $\lambda = (\log\rho + b)/C$ and a pair of nodes separated by an Euclidean distance x are directly connected with probability $g(x)$, independent of other connections. Results in [10] are derived for $\mathcal{G}\left(\mathcal{X}_\lambda, g, A_{\frac{1}{r_\rho}}\right)$. By proper scaling, it is straightforward to extend the results for $\mathcal{G}\left(\mathcal{X}_\lambda, g, A_{\frac{1}{r_\rho}}\right)$ to $\mathcal{G}\left(\mathcal{X}_\rho, g_{r_\rho}, A\right)$. Therefore we ignore the difference.

due to the truncation effect [10], it appears difficult to establish such a conclusion using [9, Theorem 6.3]. Indeed differently from $\mathcal{G}\left(\mathcal{X}_\rho, g, \Re^2\right)$ in which an unbounded component may exist for a finite ρ, it can be easily shown that for any finite ρ, $\Pr\left(|\mathcal{X}_\rho| < \infty\right) = 1$, i.e. the total number of nodes in $\mathcal{G}\left(\mathcal{X}_\rho, g_{r_\rho}, A\right)$ is almost surely finite. It then follows that for any finite ρ almost surely there is no unbounded component in $\mathcal{G}\left(\mathcal{X}_\rho, g_{r_\rho}, A\right)$.

In this paper, we solve the above conceptual difficulty involving use of the term "unbounded component" by considering the number of components in $\mathcal{G}\left(\mathcal{X}_\rho, g_{r_\rho}, A\right)$ of order greater than M, denoted by $\xi_{>M}$, where M is an arbitrarily large positive integer. We then show that $\lim_{M\to\infty}\lim_{\rho\to\infty}\Pr(\xi_{>M} = 1) = 1$. The analytical result is summarized in the following theorem, which forms a further major contribution of this paper:

Theorem 6. *As $\rho \to \infty$,* a.a.s. *the number of unbounded components in $\mathcal{G}\left(\mathcal{X}_\rho, g_{r_\rho}, A\right)$ is one.*

Proof: See Appendix II ■

Remark 7. Proof of the type of results in Theorem 6 usually requires some complicated geometric analysis. Particularly the proof of Lemma 15 in Appendix II, which forms a foundation of the proof of Theorem 6, needs sophisticated geometric analysis. In this paper, we omitted the proof of Lemma 15 because the proof is exactly the same as the proof of Theorem 2, which in turn relies on some results established in [10]. We refer interested readers to the proof of Theorem 1 in [10] for techniques on handling geometric obstacles involved in analyzing the boundary effect and to the proof of Theorem 4 in [10] for techniques on handling geometric obstacles involved in analyzing the number of components in $\mathcal{G}\left(\mathcal{X}_\rho, g_{r_\rho}, A\right)$.

An implication of Theorem 6 is that for an arbitrarily small positive constant ε, there exists large positive constants M_0 and ρ_0 such that for all $M > M_0$ and $\rho > \rho_0$, $\Pr(\xi_{>M} = 1) > 1 - \varepsilon$. From (61) in Appendix II, it can further be concluded that for a particular positive integer M and an arbitrarily small positive constant ε, there exists ρ_0 such that for all $\rho > \rho_0$,

$$\Pr(\xi_{>M} = 1) > 1 - \frac{e^{-(M+1)b}}{(M+1)!} - \varepsilon \tag{8}$$

The following corollary can be obtained from [10, Theorem 4] and Theorem 6:

Corollary 8. *As $\rho \to \infty$, a.a.s. $\mathcal{G}\left(\mathcal{X}_\rho, g_{r_\rho}, A\right)$ forms a connected network iff there is no isolated node in it.*

Proof: Let ξ be the total number of components in $\mathcal{G}\left(\mathcal{X}_\rho, g_{r_\rho}, A\right)$. It is clear that $\xi = \xi_1 + \sum_{k=2}^{M}\xi_k + \xi_{>M}$, where ξ_k is the number of components of order k. Noting that $\xi = 1$ iff $\mathcal{G}\left(\mathcal{X}_\rho, g_{r_\rho}, A\right)$ forms a connected network, it suffices to show that $\lim_{\rho\to\infty}\Pr\left(\xi = 1|\xi_1 = 0\right) = 1$. We observe that

$$\begin{aligned}
&\Pr\left(\xi = 1, \xi_1 = 0\right) \\
&\geq \Pr(\xi_1 = 0, \sum_{k=2}^{M}\xi_k = 0, \xi_{>M} = 1) \\
&= \Pr(\xi_1 = 0) - (\Pr(\overline{\sum_{k=2}^{M}\xi_k = 0}) + \Pr(\overline{\xi_{>M} = 1}))
\end{aligned} \tag{9}$$

where in (9) $\overline{\xi_{>M} = 1}$ represents the complement of the event $\xi_{>M} = 1$ and (9) results as a consequence of the union bound. Further note that (9) is valid for *any* value of M and that $\Pr\left(\xi_1 = 0\right)$ converges to a *non-zero* constant $e^{-e^{-b}}$ as $\rho \to \infty$ (Theorem 2). Using the above results, [10, Theorem 4] which showed that $\lim_{\rho\to\infty}\Pr(\sum_{k=2}^{M}\xi_k = 0) = 1$, and (8), and following a few simple steps (omitted), it can be shown that for an arbitrarily small positive constant ε, by choosing M to be sufficiently large, there exists ρ_0 such that for all $\rho > \rho_0$, $\Pr\left(\xi = 1|\xi_1 = 0\right) > 1 - \varepsilon$. ■

As an easy consequence of Theorem 2 and Corollary 8, the following theorem can be established:

Theorem 9. *As $\rho \to \infty$, the probability that $\mathcal{G}\left(\mathcal{X}_\rho, g_{r_\rho}, A\right)$ forms a connected network converges to $e^{-e^{-b}}$.*

Using the above theorem and a similar analysis as that leading to Theorem 4 and Corollary 5, the following theorem on the sufficient and necessary condition for $\mathcal{G}\left(\mathcal{X}_\rho, g_{r_\rho}, A\right)$ to be *a.a.s.* connected can be obtained:

Theorem 10. *As $\rho \to \infty$, $\mathcal{G}\left(\mathcal{X}_\rho, g_{r_\rho}, A\right)$ is* a.a.s. *connected iff $b \to \infty$; $\mathcal{G}\left(\mathcal{X}_\rho, g_{r_\rho}, A\right)$ is* a.a.s. *disconnected iff $b \to -\infty$.*

V. Conclusion and Further Work

Following the seminal work of Penrose [3], [5] and Gupta and Kumar [2] on the asymptotic connectivity of large-scale random networks with Poisson node distribution and under the unit disk model, there is general expectation that there is a range of connection functions for which the above results [2], [3], [5] obtained assuming the unit disk model can carry over. However, for quite a long time, both the asymptotic laws that the network should follow and the conditions on the connection function required for the network to be *a.a.s.* connected under a more generic setting have been unknown. In this paper, we filled in the gaps by providing the sufficient and necessary condition for a network with nodes Poissonly distributed on a unit square and following a generic random connection model to be *a.a.s.* connected as $\rho \to \infty$. The conditions on the connection function required in order for the above network to be *a.a.s.* connected were also provided. Therefore, the results in the paper constitute a significant advance of the earlier work by Penrose [3], [5] and Gupta and Kumar [2] from the unit disk model to the more generic random connection model and bring models addressed by theoretical research closer to reality.

However, there remain significant challenges ahead. The results in this paper rely on three main assumptions: a) the connection function g is isotropic, b) the random events underpinning generation of a connection are independent, c) nodes are Poissonly distributed. We conjecture that assumption a) is not a critical assumption, i.e. under some mild conditions, e.g. nodes are independently and randomly oriented, assumption a) can be removed while our results are still valid. It is part of our future work plan to validate the conjecture. Our results however critically rely on assumption b), which is

not necessarily valid in some real networks due to channel correlation and interference, where the latter effect makes the connection between a pair of nodes dependent on the locations and activities of other nearby nodes. In [19] we have done some preliminary work on network connectivity considering the impact of interference. The work essentially uses a decoupling approach to solve the challenges of connection correlation caused by interference and suggests that when some realistic constraints are considered, i.e. carrier-sensing, the connectivity results will be very close to those obtained under a unit disk model. This conclusion is in contrast with that [20] obtained under an ALOHA multiple-access protocol. A more thorough investigation is yet to be done. The major obstacle in dealing with the impact of channel correlation is that there is no widely accepted model in the wireless communication community capturing the impact of channel correlation on connections. Finally, it is a logical move after our work to consider connectivity of networks with nodes following a generic distribution other than Poisson. It is part of our future work plan to tackle the problem.

Appendix I: Proof of Theorem 1

Our proof relies on the use of the Chen-Stein bound [16], [17]. We first establish some preliminary results that allow us to use the Chen-Stein bound for the analysis of number of isolated nodes in $\mathcal{G}^T\left(\mathcal{X}_\rho, g_{r_\rho}, A\right)$.

Divide the unit torus into m^2 non-overlapping squares each with size $\frac{1}{m^2}$. Denote the i_m^{th} square by A_{i_m}. Define two sets of indicator random variables $J_{i_m}^T$ and $I_{i_m}^T$ with $i_m \in \Gamma_m \triangleq \{1,\ldots m^2\}$, where $J_{i_m}^T=1$ iff there exists exactly one node in A_{i_m}, otherwise $J_{i_m}^T=0$; $I_{i_m}^T=1$ iff there is exactly one node in A_{i_m} *and* that node is isolated, $I_{i_m}^T=0$ otherwise. Obviously $J_{i_m}^T$ is independent of $J_{j_m}^T, j_m \in \Gamma_m \backslash \{i_m\}$. Denote the center of $A_{i_m}^T$ by $\boldsymbol{x}_{i_m}$ and without loss of generality we assume that when $J_{i_m}^T=1$, the associated node in A_{i_m} is at $\boldsymbol{x}_{i_m}$[4]. Observe that for any fixed m, the values of $\Pr\left(I_{i_m}^T=1\right)$ and $\Pr\left(J_{i_m}^T=1\right)$ do not depend on the particular index i_m on a torus. However both the set of indices Γ_m and a particular index i_m depend on m. As m changes, the square associated with $I_{i_m}^T$ and $J_{i_m}^T$ also changes.

Remark 11. In this paper, we are only interested in the limiting values of various parameters associated with a sub-square as $m\to\infty$. Also because of the consideration of a torus, the value of a particular index i_m does not affect the discussion of the associated parameters, i.e. these parameters $I_{i_m}^T$ and $J_{i_m}^T$ do not depend on i_m. Therefore in the following, we omit some straightforward discussions on the convergence of various parameters, e.g. i_m, $\boldsymbol{x}_{i_m}$, $I_{i_m}^T$ and $J_{i_m}^T$, as $m\to\infty$.

Without causing ambiguity, we drop the explicit dependence on m in our notations for convenience. As an easy consequence of the Poisson node distribution, $\Pr(J_i^T=1)\sim_m \rho/m^2$. Using [9, Proposition 1.3], $\Pr(I_i^T=1)=\Pr(I_i^T=1|J_i^T=1)\Pr(J_i^T=1)$ and the property of a torus (see also [10, Lemma 1]), it can be shown that

$$\begin{aligned}\Pr(I_i^T=1) &\sim_m \frac{\rho}{m^2}e^{-\int_A \rho g(\frac{\|\boldsymbol{x}-\boldsymbol{x}_i\|^T}{r_\rho})d\boldsymbol{x}}\\ &= \frac{\rho}{m^2}e^{-\int_A \rho g(\frac{\|\boldsymbol{x}\|^T}{r_\rho})d\boldsymbol{x}} \end{aligned}\tag{10}$$

Now consider the event $I_i^T I_j^T=1, i\neq j$, conditioned on the event that $J_i^T J_j^T=1$, meaning that both nodes having been placed inside A_i and A_j respectively are isolated. Following the same steps leading to (10), it can be shown that

$$\begin{aligned}&\lim_{m\to\infty}\Pr(I_i^T I_j^T=1|J_i^T J_j^T=1)\\ =&(1-g(\frac{\|\boldsymbol{x}_i-\boldsymbol{x}_j\|^T}{r_\rho}))\exp[-\int_A\rho(g(\frac{\|\boldsymbol{x}-\boldsymbol{x}_i\|^T}{r_\rho})\\ &+g(\frac{\|\boldsymbol{x}-\boldsymbol{x}_j\|^T}{r_\rho})-g(\frac{\|\boldsymbol{x}-\boldsymbol{x}_i\|^T}{r_\rho})g(\frac{\|\boldsymbol{x}-\boldsymbol{x}_j\|^T}{r_\rho}))d\boldsymbol{x}]\end{aligned}\tag{11}$$

where the term $(1-g(\frac{\|\boldsymbol{x}_i-\boldsymbol{x}_j\|^T}{r_\rho}))$ is due to the requirement that the two nodes located inside A_i and A_j cannot be directly connected given that they are both isolated nodes. Observe also that $\Pr(I_i^T I_j^T=1)=\Pr(J_i^T J_j^T=1)\Pr(I_j^T I_j^T=1|J_i^T J_j^T=1)$. Now using the above equation, (10) and (11), it can be established that

$$\begin{aligned}&\frac{\Pr(I_i^T I_j^T=1)}{\Pr(I_i^T=1)\Pr(I_j^T=1)}\\ \sim_m&(1-g(\frac{\|\boldsymbol{x}_i-\boldsymbol{x}_j\|^T}{r_\rho}))e^{\int_A\rho g(\frac{\|\boldsymbol{x}-\boldsymbol{x}_i\|^T}{r_\rho})g(\frac{\|\boldsymbol{x}-\boldsymbol{x}_j\|^T}{r_\rho})d\boldsymbol{x}}\end{aligned}\tag{12}$$

Now we are ready to use the Chen-Stein bound to prove Theorem 1. Particularly, we will show using the Chen-Stein bound that

$$W^T=\lim_{m\to\infty}\sum_{i\in\Gamma_m} I_i^T \tag{13}$$

asymptotically converges to a Poisson distribution with mean e^{-b} as $\rho\to\infty$.

The following theorem gives a formal statement of the Chen-Stein bound:

Theorem 12. *[17, Theorem 1.A] For a set of indicator random variables $I_i,\ i\in\Gamma$, define $W\triangleq\sum_{i\in\Gamma}I_i$, $p_i\triangleq E\left(I_i\right)$ and $\eta\triangleq E\left(W\right)$. For any choice of the index set $\Gamma_{s,i}\subset\Gamma$, $\Gamma_{s,i}\cap\{i\}=\{$ï¿œ$\}$,*

$$\begin{aligned}&d_{TV}\left(\mathcal{L}\left(W\right),Po\left(\eta\right)\right)\\ \leq&\sum_{i\in\Gamma}[(p_i^2+p_iE(\sum_{j\in\Gamma_{s,i}}I_j))]\min(1,\frac{1}{\eta})\\ +&\sum_{i\in\Gamma}E(I_i\sum_{j\in\Gamma_{s,i}}I_j)\min(1,\frac{1}{\eta})\\ +&\sum_{i\in\Gamma}E|E\{I_i|(I_j,j\in\Gamma_{w,i})\}-p_i|\min(1,\frac{1}{\eta})\end{aligned}$$

where $\mathcal{L}\left(W\right)$ denotes the distribution of W, $Po\left(\eta\right)$ denotes a Poisson distribution with mean η, $\Gamma_{w,i}=\Gamma\backslash\left\{\Gamma_{s,i}\cup\{i\}\right\}$ and d_{TV} denotes the total variation distance. The total variation distance between two probability distributions α and β on $\mathbb{Z}^+$ is given by $d_{TV}\left(\alpha,\beta\right)\triangleq\sup\left\{\left|\alpha\left(A\right)-\beta\left(A\right)\right|:A\subset\mathbb{Z}^+\right\}$.

[4] In this paper we are mainly concerned with the case that $m\to\infty$, i.e. the size of the square is vanishingly small. Therefore the actual position of the node in the square is not important.

For convenience, we separate the bound in Theorem 12 into three terms $b_1 \min(1, \frac{1}{\eta})$, $b_2 \min(1, \frac{1}{\eta})$ and $b_3 \min(1, \frac{1}{\eta})$ where

$$b_1 \triangleq \sum_{i\in\Gamma}[(p_i^2 + p_i E(\sum_{j\in\Gamma_{s,i}} I_j))] \tag{14}$$

$$b_2 \triangleq \sum_{i\in\Gamma} E(I_i \sum_{j\in\Gamma_{s,i}} I_j) \tag{15}$$

$$b_3 \triangleq \sum_{i\in\Gamma} E|E\{I_i|(I_j, j\in\Gamma_{w,i})\} - p_i| \tag{16}$$

The set of indices $\Gamma_{s,i}$ is often chosen to contain all those j, other than i, for which I_j is "strongly" dependent on I_i and the set $\Gamma_{w,i}$ often contains all other indices apart from i for which I_j is at most "weakly" dependent on I_i [16].

Remark 13. A main challenge in using the Chen-Stein bound to prove Theorem 1 is that under the random connection model, the two events I_i and I_j may be correlated even when $\boldsymbol{x}_i$ and $\boldsymbol{x}_j$ are separated by a very large Euclidean distance. Therefore the dependence structure is global, which significantly increases the complexity of the analysis. In comparison, in applications where the dependence structure is local, by a suitable choice of $\Gamma_{s,i}$ the b_3 term can be easily made to be 0 and the evaluation of the b_1 and b_2 terms involves the computation of the first two moments of W only, which can often be achieved relatively easily. An example is a random geometric network under the unit disk model. If $\Gamma_{s,i}$ is chosen to be a neighborhood of i containing indices of all nodes whose distance to node i is less than or equal to twice the transmission range, the b_3 term is easily shown to be 0. It can then be readily shown that the b_1 and b_2 terms approach 0 as the neighborhood size of a node becomes vanishingly small compared to the overall network size as $\rho \to \infty$ [14]. However this is certainly not the case for the random connection model.

Remark 14. The key idea involved using the Chen-Stein bound to prove Theorem 1 is constructing a neighborhood of a node, i.e. $\Gamma_{s,i}$ in Theorem 12, such that a) the size of the neighborhood becomes vanishingly small compared with A as $\rho \to \infty$. This is required for the b_1 and b_2 terms to approach 0 as $\rho \to \infty$; b) *a.a.s.* the neighborhood contains all nodes that may have a direct connection with the node. This is required for the b_3 term to approach 0 as $\rho \to \infty$. Such a neighborhood is defined in the next paragraph.

Let $D^T(\boldsymbol{x}_i, r) \triangleq \{\boldsymbol{x} \in A : \|\boldsymbol{x} - \boldsymbol{x}_i\|^T \le r\}$ and when $\boldsymbol{x}_i$ is not within r of the border of A, $D^T(\boldsymbol{x}_i, r)$ becomes the same as $D(\boldsymbol{x}_i, r)$ where $D(\boldsymbol{x}_i, r) \triangleq \{\boldsymbol{x} \in A : \|\boldsymbol{x} - \boldsymbol{x}_i\| \le r\}$. Further define the neighborhood of an index $i \in \Gamma$ as $\Gamma_{s,i} \triangleq \{j : \boldsymbol{x}_j \in D^T(\boldsymbol{x}_i, 2r_\rho^{1-\epsilon})\}\backslash\{i\}$ and define the non-neighborhood of the index i as $\Gamma_{w,i} \triangleq \{j : \boldsymbol{x}_j \notin D^T(\boldsymbol{x}_i, 2r_\rho^{1-\epsilon})\}$ where ϵ is a small positive constant and $\epsilon \in (0, \frac{1}{2})$. It can be shown that

$$|\Gamma_{s,i}| = m^2 4\pi r_\rho^{2-2\epsilon} + o_m(m^2 4\pi r_\rho^{2-2\epsilon}) \tag{17}$$

Note that in Theorem 12, $p_i = E(I_i^T)$ and $E(I_i^T)$ has been given in (10). Further, as an easy consequence of (13) and [10, Lemma 1] which showed that

$$\lim_{\rho\to\infty} E(W^T) = \lim_{\rho\to\infty} \rho e^{-\int_A \rho g(\frac{\|\boldsymbol{x}\|^T}{r_\rho})d\boldsymbol{x}} = e^{-b} \tag{18}$$

$\lim_{\rho\to\infty}\lim_{m\to\infty}\eta = e^{-b}$.

Using (10), $p_i = E(I_i^T)$ and (18), it follows that

$$\lim_{m\to\infty} m^2 p_i = \rho e^{-\int_A \rho g(\frac{\|\boldsymbol{x}\|^T}{r_\rho})d\boldsymbol{x}} \tag{19}$$

$$\lim_{\rho\to\infty}\lim_{m\to\infty} m^2 p_i = e^{-b} \tag{20}$$

Next we shall evaluate the b_1, b_2 and b_3 terms in the following three subsections separately and show that all three terms converge to 0 as $\rho \to \infty$.

A. An Evaluation of the b_1 Term

It can be shown that (following the equation, detailed explanations are given)

$$\begin{aligned}
&\lim_{\rho\to\infty}\lim_{m\to\infty}\sum_{i\in\Gamma}(p_i^2 + p_i E(\sum_{j\in\Gamma_{s,i}} I_j^T)) \\
&= \lim_{\rho\to\infty}\lim_{m\to\infty} m^2 p_i E(\sum_{j\in\Gamma_{s,i}\cup\{i\}} I_j^T) \\
&= \lim_{\rho\to\infty}\lim_{m\to\infty} (m^2 p_i)^2 4\pi r_\rho^{2-2\epsilon} \qquad (21)\\
&= \lim_{\rho\to\infty} 4\pi(\rho e^{-\int_A \rho g(\frac{\|\boldsymbol{x}-\boldsymbol{x}_i\|^T}{r_\rho})d\boldsymbol{x}})^2(\frac{\log\rho + b}{C\rho})^{1-\epsilon} \qquad (22)\\
&= 4\pi e^{-2b}\lim_{\rho\to\infty}(\frac{\log\rho + b}{C\rho})^{1-\epsilon} = 0 \qquad (23)
\end{aligned}$$

where (17) is used in obtaining (21); (1) and (19) are used in obtaining (22); and (18) and (20) are used in obtaining (23). Therefore $\lim_{\rho\to\infty}\lim_{m\to\infty} b_1 = 0$.

B. An Evaluation of the b_2 Term

For the b_2 term, assume that ρ is sufficiently large such that $\frac{1}{2r_\rho} >> 2r_\rho^{-\epsilon}$ and let $A_{\frac{1}{r_\rho}} = [-\frac{1}{2r_\rho}, \frac{1}{2r_\rho}]^2$. Using (11) in the first step; and first using some translation and scaling operations and then using (7) in the last step, equation (24) can be obtained.

Letting $\lambda \triangleq \frac{\log\rho + b}{C}$ for convenience, noting that (using (7) and (3))

$$\lim_{\rho\to\infty}\int_{A_{\frac{1}{r_\rho}}} g(\|\boldsymbol{x}\|^T)d\boldsymbol{x} = \lim_{\rho\to\infty}\int_{A_{\frac{1}{r_\rho}}} g(\|\boldsymbol{x}-\boldsymbol{y}\|^T)d\boldsymbol{x} = C$$

and that $1 - g(\|\boldsymbol{y}\|^T) \le 1$, it can further be shown following (24) that as $\rho \to \infty$,

$$\begin{aligned}
&\lim_{\rho\to\infty}\lim_{m\to\infty}\sum_{i\in\Gamma} E(I_i^T \sum_{j\in\Gamma_{s,i}} I_j^T) \\
&\le e^{-2b}\lim_{\rho\to\infty}\frac{\lambda}{\rho}\int_{D(\boldsymbol{0}, 2r_\rho^{-\epsilon})} e^{\lambda\int_{A_{\frac{1}{r_\rho}}} g(\|\boldsymbol{x}\|^T)g(\|\boldsymbol{x}-\boldsymbol{y}\|^T)d\boldsymbol{x}} d\boldsymbol{y}
\end{aligned} \tag{25}$$

In the following paragraphs, we will show that the right hand side of (25) converges to 0 as $\rho \to \infty$. Using (2) and (3), we assert that there exists a positive constant r such that $g(r^-)(1 - g(r^+)) > 0$ where $g(r^-) \triangleq \lim_{x\to r^-} g(x)$ and $g(r^+) \triangleq \lim_{x\to r^+} g(x)$. Indeed if g is a continuous function, any positive constant r with $g(r) > 0$ satisfies the requirement; if g is a discontinuous function, e.g. a unit

$$
\begin{aligned}
&\lim_{m\to\infty}\sum_{i\in\Gamma}E(I_i^T\sum_{j\in\Gamma_{s,i}}I_j^T)\\
=&\lim_{m\to\infty}\frac{\rho^2}{m^2}\sum_{j\in\Gamma_{s,i}}\{(1-g(\left\|\frac{\boldsymbol{x}_i-\boldsymbol{x}_j}{r_\rho}\right\|^T)\exp[-\int_A\rho(g(\left\|\frac{\boldsymbol{x}-\boldsymbol{x}_i}{r_\rho}\right\|^T)+g(\left\|\frac{\boldsymbol{x}-\boldsymbol{x}_j}{r_\rho}\right\|^T)-g(\left\|\frac{\boldsymbol{x}-\boldsymbol{x}_i}{r_\rho}\right\|^T)g(\left\|\frac{\boldsymbol{x}-\boldsymbol{x}_j}{r_\rho}\right\|^T))]\}d\boldsymbol{x}\\
=&\rho^2\int_{D^T\left(\boldsymbol{x}_i,2r_\rho^{1-\epsilon}\right)}\{(1-g(\frac{\|\boldsymbol{x}_i-\boldsymbol{y}\|^T}{r_\rho}))\exp[-\int_A\rho(g(\left\|\frac{\boldsymbol{x}-\boldsymbol{x}_i}{r_\rho}\right\|^T)+g(\left\|\frac{\boldsymbol{x}-\boldsymbol{y}}{r_\rho}\right\|^T)-g(\left\|\frac{\boldsymbol{x}-\boldsymbol{x}_i}{r_\rho}\right\|^T)g(\left\|\frac{\boldsymbol{x}-\boldsymbol{y}}{r_\rho}\right\|^T))d\boldsymbol{x}]\}d\boldsymbol{y}\\
=&\rho^2r_\rho^2\int_{D\left(\mathbf{0},2r_\rho^{-\epsilon}\right)}\{(1-g(\|\boldsymbol{y}\|^T))\exp[-\rho r_\rho^2\int_{A_{\frac{1}{r_\rho}}}(g(\|\boldsymbol{x}\|^T)+g(\|\boldsymbol{x}-\boldsymbol{y}\|^T)-g(\|\boldsymbol{x}\|^T)g(\|\boldsymbol{x}-\boldsymbol{y}\|^T))d\boldsymbol{x}]\}d\boldsymbol{y}
\end{aligned}
\tag{24}
$$

disk model, by choosing r to be the transmission range, $g\left(r^-\right)\left(1-g\left(r^+\right)\right)=1$.

In the following discussion we assume that ρ is sufficiently large such that $\frac{1}{2r_\rho} >> 2r_\rho^{-\epsilon} >> r$. It can be shown using (2), (3) and (7) that for $\boldsymbol{y}\in D(\mathbf{0},2r_\rho^{-\epsilon})$,

$$
\begin{aligned}
&\int_{A_{\frac{1}{r_\rho}}}g(\|\boldsymbol{x}\|^T)g(\|\boldsymbol{x}-\boldsymbol{y}\|^T)d\boldsymbol{x}\\
&\le\int_{\Re^2}g(\|\boldsymbol{x}\|)g(\|\boldsymbol{x}-\boldsymbol{y}\|)d\boldsymbol{x}\\
&=C-\int_{\Re^2}g(\|\boldsymbol{x}\|)(1-g(\|\boldsymbol{x}-\boldsymbol{y}\|))d\boldsymbol{x}\\
&\le C-\int_{D(\mathbf{0},r)\backslash D(\boldsymbol{y},r)}g(\|\boldsymbol{x}\|)(1-g(\|\boldsymbol{x}-\boldsymbol{y}\|))d\boldsymbol{x}\\
&\le C-g\left(r^-\right)\left(1-g\left(r^+\right)\right)|D(\mathbf{0},r)\backslash D(\boldsymbol{y},r)|
\end{aligned}
\tag{26}
$$

Let $f\left(x\right)\triangleq\pi r^2-2r^2\arcsin(\sqrt{1-x^2/(4r^2)})+rx\sqrt{1-x^2/(4r^2)}$. Using some simple geometric analysis, it can be shown that

- when $\|\boldsymbol{y}\|>2r$, $|D(\mathbf{0},r)\backslash D(\boldsymbol{y},r)|=\pi r^2$; and
- when $\|\boldsymbol{y}\|\le 2r$, $|D(\mathbf{0},r)\backslash D(\boldsymbol{y},r)|=f(\|\boldsymbol{y}\|)$.

Further, using the definition of $f\left(x\right)$, it can be shown that

- when $\|\boldsymbol{y}\|\le r$, $|D(\mathbf{0},r)\backslash D(\boldsymbol{y},r)|\ge\sqrt{3}r\,\|\boldsymbol{y}\|$; and
- when $\|\boldsymbol{y}\|>r$, $|D(\mathbf{0},r)\backslash D(\boldsymbol{y},r)|\ge(\frac{\pi}{3}+\frac{\sqrt{3}}{2})r^2$.

For convenience, let $c_1\triangleq g\left(r^-\right)\left(1-g\left(r^+\right)\right)\sqrt{3}r$ and $c_2\triangleq g\left(r^-\right)\left(1-g\left(r^+\right)\right)\left(\frac{\pi}{3}+\frac{\sqrt{3}}{2}\right)r^2$. Noting that $g\left(r^-\right)(1-g\left(r^+\right))>0$, c_1 and c_2 are positive constants, independent of both $\boldsymbol{y}$ and ρ.

As a result of (26) and the above inequalities on $|D(\mathbf{0},r)\backslash D(\boldsymbol{y},r)|$, it follows that

$$
\begin{aligned}
&\lim_{\rho\to\infty}\frac{\lambda}{\rho}\int_{D(\mathbf{0},2r_\rho^{-\epsilon})}e^{\lambda\int_{A_{\frac{1}{r_\rho}}}g(\|\boldsymbol{x}\|^T)g(\|\boldsymbol{x}-\boldsymbol{y}\|^T)d\boldsymbol{x}}d\boldsymbol{y}\\
&\le\lim_{\rho\to\infty}\frac{\lambda}{\rho}\int_{D(\mathbf{0},r)}e^{\lambda(C-c_1\|\boldsymbol{y}\|)}d\boldsymbol{y}\\
&+\lim_{\rho\to\infty}\frac{\lambda}{\rho}\int_{D\left(\mathbf{0},2r_\rho^{-\epsilon}\right)\backslash D(\mathbf{0},r)}e^{\lambda(C-c_2)}d\boldsymbol{y}
\end{aligned}
\tag{27}
$$

For the first summand in the above equation, it can be shown that:

$$
\begin{aligned}
&\lim_{\rho\to\infty}\frac{\lambda}{\rho}\int_{D(\mathbf{0},r)}e^{\lambda(C-c_1\|\boldsymbol{y}\|)}d\boldsymbol{y}\\
&=\lim_{\rho\to\infty}\frac{\log\rho+b}{C\rho}\int_0^r2\pi ye^{\frac{\log\rho+b}{C}(C-c_1y)}dy=0
\end{aligned}
\tag{28}
$$

For the second summand in (27), by choosing $\varepsilon<\frac{c_2}{C}$ and using (1), it follows that

$$
\begin{aligned}
&\lim_{\rho\to\infty}\frac{\lambda}{\rho}\int_{D(\mathbf{0},2r_\rho^{-\epsilon})\backslash D(\mathbf{0},r)}e^{\lambda(C-c_2)}d\boldsymbol{y}\\
&=\lim_{\rho\to\infty}\frac{e^{b(1-\frac{c_2}{C})}}{C}\times\frac{\log\rho+b}{\rho^{\frac{c_2}{C}}}\times\pi(4r_\rho^{-2\epsilon}-r^2)=0
\end{aligned}
\tag{29}
$$

Combining (27), (28) and (29), it follows that

$$
\lim_{\rho\to\infty}\frac{\lambda}{\rho}\int_{D(\mathbf{0},2r_\rho^{-\epsilon})}e^{\lambda\int_{A_{\frac{1}{r_\rho}}}g(\|\boldsymbol{x}\|^T)g(\|\boldsymbol{x}-\boldsymbol{y}\|^T)d\boldsymbol{x}}d\boldsymbol{y}=0
\tag{30}
$$

As a result of (25) and the above equation: $\lim_{\rho\to\infty}\lim_{m\to\infty}b_2=0$.

C. An Evaluation of the b_3 Term

We first obtain an analytical expression of the term $E\{I_i|(I_j,j\in\Gamma_{w,i})\}$ in b_3. Using the same procedure that results in (12), it can be obtained that (for convenience we use g_i for $g(\frac{\|\boldsymbol{x}-\boldsymbol{x}_i\|^T}{r_\rho})$ and use g_{ij} for $g(\frac{\|\boldsymbol{x}_i-\boldsymbol{x}_j\|^T}{r_\rho})$ in the following equation):

$$
\begin{aligned}
&\lim_{m\to\infty}\frac{Pr(I_i^T=1,I_j^T=1,I_k^T=0)}{Pr(I_i^T=1)Pr(I_j^T=1,I_k^T=0)}\\
&=\lim_{m\to\infty}\frac{Pr(I_i^T=1,I_j^T=1)-Pr(I_i^T=1,I_j^T=1,I_k^T=1)}{Pr(I_i^T=1)(Pr(I_j^T=1)-Pr(I_j^T=1,I_k^T=1))}\\
&\sim_m(1-g_{ij})\,e^{\int_A\rho g_ig_jd\boldsymbol{x}}\\
&\quad\times\frac{1-\frac{\rho}{m^2}(1-g_{ik})(1-g_{kj})e^{-\int_A\rho(g_k-g_ig_k-g_kg_j+g_ig_jg_k)d\boldsymbol{x}}}{1-\frac{\rho}{m^2}(1-g_{kj})e^{-\int_A\rho(g_k-g_kg_j)d\boldsymbol{x}}}
\end{aligned}
\tag{31}
$$

Using (2), (3), (4) and (7), it can be shown that when $j\in\Gamma_{w,i}$ (or equivalently $\|\boldsymbol{x}_i-\boldsymbol{x}_j\|^T>2r_\rho^{1-\varepsilon}$), the integrals of some higher order terms inside the exponential function in (31) satisfy:

$$
\begin{aligned}
&\int_A\rho g(\frac{\|\boldsymbol{x}-\boldsymbol{x}_i\|^T}{r_\rho})g(\frac{\|\boldsymbol{x}-\boldsymbol{x}_j\|^T}{r_\rho})d\boldsymbol{x}\\
&=\int_{D^T(\boldsymbol{x}_i,r_\rho^{1-\varepsilon})}\rho g(\frac{\|\boldsymbol{x}-\boldsymbol{x}_i\|^T}{r_\rho})g(\frac{\|\boldsymbol{x}-\boldsymbol{x}_j\|^T}{r_\rho})d\boldsymbol{x}
\end{aligned}
$$

$$+\int_{A\setminus D^T(\boldsymbol{x}_i,r_\rho^{1-\varepsilon})}\rho g(\frac{\|\boldsymbol{x}-\boldsymbol{x}_i\|^T}{r_\rho})g(\frac{\|\boldsymbol{x}-\boldsymbol{x}_j\|^T}{r_\rho})d\boldsymbol{x}$$
$$\leq 2C\rho r_\rho^2 g(r_\rho^{-\varepsilon}) \sim_\rho o_\rho(1)$$

Note also that $g_{ik} = g(\frac{\|\boldsymbol{x}_i-\boldsymbol{x}_k\|^T}{r_\rho}) = o_\rho(1)$ for $k \in \Gamma_{w,i}$. Using the above equations and (12), it can be further shown following (31) that when $j,k \in \Gamma_{w,i}$.

$$\lim_{\rho\to\infty}\lim_{m\to\infty}\frac{Pr(I_i^T=1,I_j^T=1,I_k^T=0)}{Pr(I_i^T=1)Pr(I_j^T=1,I_k^T=0)}$$
$$=\lim_{\rho\to\infty}\lim_{m\to\infty}\frac{Pr(I_i^T=1,I_j^T=1)}{Pr(I_i^T=1)Pr(I_j^T=1)} \qquad (32)$$

Equation (32) shows that the impact of those events, whose associated indicator random variables $I_k^T=0, k\in\Gamma_{w,i}$, on the event $I_i^T=1$ is asymptotically vanishingly small, hence can be ignored. Denote by Γ_i a random set of indices containing all indices j where $j\in\Gamma_{w,i}$ *and* $I_j=1$, i.e. the node in question is also isolated, and denote by γ_i an instance of Γ_i. Define $n \triangleq |\gamma_i|$. Following the same procedure that results in (32), it can be established that (with some verbose but straightforward discussions omitted)

$$\lim_{\rho\to\infty}\lim_{m\to\infty}\frac{E\{I_i^T|(I_j^T, j\in\Gamma_{w,i})\}}{\frac{\rho}{m^2}}$$
$$=\lim_{\rho\to\infty}\lim_{m\to\infty}\frac{E\{I_i^T|(I_j^T=1, j\in\gamma_i)\}}{\frac{\rho}{m^2}}$$
$$=\lim_{\rho\to\infty}E[e^{-\int_A\rho g(\frac{\|\boldsymbol{x}-\boldsymbol{x}_i\|^T}{r_\rho})\prod_{j\in\gamma_i}(1-g(\frac{\|\boldsymbol{x}-\boldsymbol{x}_j\|^T}{r_\rho}))d\boldsymbol{x}}$$
$$\times\prod_{j\in\gamma_i}(1-g(\frac{\|\boldsymbol{x}_i-\boldsymbol{x}_j\|^T}{r_\rho}))] \qquad (33)$$

Equation (33) gives an analytical expression of the term $E\{I_i^T|(I_j^T, j\in\Gamma_{w,i})\}$. To solve the challenges associated with handling the absolute value term in b_3, viz. $|E\{I_i^T|(I_j^T, j\in\Gamma_{w,i})\}-p_i|$, we further obtain an upper and a lower bound of $I_i^T|(I_j^T, j\in\Gamma_{w,i})$, which allows us to remove the absolute value sign in the further analysis of b_3.

Note that $\boldsymbol{x}_i$ and $\boldsymbol{x}_j, j\in\Gamma_{w,i}$ is separated by a distance not smaller than $2r_\rho^{-\epsilon}$. Using (2), a lower bound on the value inside the expectation operator in (33) is given by

$$B_{L,i} \triangleq (1-g(2r_\rho^{-\epsilon}))^n e^{-\int_A\rho g(\frac{\|\boldsymbol{x}-\boldsymbol{x}_i\|^T}{r_\rho})d\boldsymbol{x}} \qquad (34)$$

An upper bound on the value inside the expectation operator in (33) is given by

$$B_{U,i}\triangleq e^{-\int_A\rho g(\frac{\|\boldsymbol{x}-\boldsymbol{x}_i\|^T}{r_\rho})\prod_{j\in\gamma_i}(1-g(\frac{\|\boldsymbol{x}-\boldsymbol{x}_j\|^T}{r_\rho}))d\boldsymbol{x}} \qquad (35)$$

Using $p_i=E(I_i^T)$ and (10), it can be shown that

$$B_{U,i}\geq\lim_{m\to\infty}\frac{m^2p_i}{\rho}\geq B_{L,i} \qquad (36)$$

Let us consider $E|E\{I_i|(I_j, j\in\Gamma_{w,i})\}-p_i|$ now. From (33), (34), (35) and (36), it is clear that

$$\lim_{\rho\to\infty}\lim_{m\to\infty}\sum_{i\in\Gamma}E|E\{I_i^T|(I_j^T, j\in\Gamma_{w,i})\}-p_i|$$
$$\in [0,\max\{\lim_{\rho\to\infty}\lim_{m\to\infty}m^2p_i-\rho E(B_{L,i}),$$
$$\lim_{\rho\to\infty}\lim_{m\to\infty}\rho E(B_{U,i})-m^2p_i\}] \qquad (37)$$

In the following we will show that both terms $\lim_{m\to\infty}m^2p_i-\rho E(B_{L,i})$ and $\lim_{m\to\infty}\rho E(B_{U,i})-m^2p_i$ in (37) approach 0 as $\rho\to\infty$. First it can be shown following (34) that

$$\lim_{m\to\infty}\rho E(B_{L,i})$$
$$\geq\lim_{m\to\infty}\rho E[(1-ng(2r_\rho^{-\epsilon}))e^{-\int_A\rho g(\frac{\|\boldsymbol{x}-\boldsymbol{x}_i\|^T}{r_\rho})d\boldsymbol{x}}]$$
$$=\lim_{m\to\infty}\rho(1-E(n)g(2r_\rho^{-\epsilon}))e^{-\int_A\rho g(\frac{\|\boldsymbol{x}-\boldsymbol{x}_i\|^T}{r_\rho})d\boldsymbol{x}} \qquad (38)$$

where $\lim_{m\to\infty}E(n)$ is the expected number of isolated nodes in $A\backslash D(\boldsymbol{x}_i, 2r_\rho^{1-\epsilon})$. In the first step of the above equation, the inequality $(1-x)^n\geq 1-nx$ for $0\leq x\leq 1$ and $n\geq 0$ is used. When $\rho\to\infty$, $r_\rho^{1-\epsilon}\to 0$ and $r_\rho^{-\epsilon}\to\infty$ therefore $\lim_{\rho\to\infty}\lim_{m\to\infty}E(n)=\lim_{\rho\to\infty}E(W^T)=e^{-b}$ is a bounded value and $\lim_{\rho\to\infty}\lim_{m\to\infty}g(2r_\rho^{-\epsilon})\to 0$, which is an immediate outcome of (4). Using (19), it then follows that

$$\lim_{\rho\to\infty}\lim_{m\to\infty}\frac{\rho E(B_{L,i})}{m^2p_i}\geq\lim_{\rho\to\infty}\lim_{m\to\infty}(1-E(n)g(2r_\rho^{-\epsilon}))=1$$

Together with (20) and (36), we conclude that

$$\lim_{\rho\to\infty}\lim_{m\to\infty}m^2p_i-\rho E(B_{L,i})=0 \qquad (39)$$

Now let us consider the second term $\lim_{m\to\infty}\rho E(B_{U,i})-m^2p_i$, it can be observed that

$$\lim_{m\to\infty}E(B_{U,i})$$
$$\leq E[e^{-\int_{D(\boldsymbol{x}_i,r_\rho^{1-\epsilon})}\rho g(\frac{\|\boldsymbol{x}-\boldsymbol{x}_i\|^T}{r_\rho})\prod_{j\in\gamma_i}(1-g(\frac{\|\boldsymbol{x}-\boldsymbol{x}_j\|^T}{r_\rho}))d\boldsymbol{x}}]$$
$$\leq\lim_{m\to\infty}E[e^{-\int_{D(\boldsymbol{x}_i,r_\rho^{1-\epsilon})}\rho g(\frac{\|\boldsymbol{x}-\boldsymbol{x}_i\|^T}{r_\rho})\prod_{j\in\gamma_i}(1-g(\frac{r_\rho^{1-\epsilon}}{r_\rho}))d\boldsymbol{x}}]$$
$$=\lim_{m\to\infty}E(e^{-(1-g(r_\rho^{-\epsilon}))^n\int_{D(\boldsymbol{x}_i,r_\rho^{1-\epsilon})}\rho g(\frac{\|\boldsymbol{x}-\boldsymbol{x}_i\|^T}{r_\rho})d\boldsymbol{x}})$$
$$\leq\lim_{m\to\infty}E(e^{-(1-ng(r_\rho^{-\epsilon}))\int_{D(\boldsymbol{x}_i,r_\rho^{1-\epsilon})}\rho g(\frac{\|\boldsymbol{x}-\boldsymbol{x}_i\|^T}{r_\rho})d\boldsymbol{x}}) \qquad (40)$$

where in the second step, the non-increasing property of g, and the fact that $\boldsymbol{x}_j$ is located in $A\backslash D(\boldsymbol{x}_i, 2r_\rho^{1-\epsilon})$ and $\boldsymbol{x}$ is located in $D(\boldsymbol{x}_i, r_\rho^{1-\epsilon})$, therefore $\|\boldsymbol{x}-\boldsymbol{x}_j\|^T\geq r_\rho^{1-\epsilon}$ is used. It can be further demonstrated that the term $\int_{D(\boldsymbol{x}_i,r_\rho^{1-\epsilon})}\rho g(\frac{\|\boldsymbol{x}-\boldsymbol{x}_i\|}{r_\rho})d\boldsymbol{x}$ in (40) have the following property:

$$\eta(\varepsilon,\rho) \triangleq \int_{D(\boldsymbol{x}_i,r_\rho^{1-\epsilon})}\rho g(\frac{\|\boldsymbol{x}-\boldsymbol{x}_i\|^T}{r_\rho})d\boldsymbol{x}$$
$$=\rho r_\rho^2\int_{D(\frac{\boldsymbol{x}_i}{r_\rho},r_\rho^{-\epsilon})}g(\|\boldsymbol{x}-\boldsymbol{x}_i/r_\rho\|^T)d\boldsymbol{x}$$
$$\leq C\rho r_\rho^2=\log\rho+b \qquad (41)$$

For the other term $ng(r_\rho^{-\epsilon})$ in (40), choosing a positive constant $\delta<2\epsilon$ and using Markov's inequality, it can be shown that $Pr(n\geq r_\rho^{-\delta})\leqslant r_\rho^\delta E(n)$. Therefore

$$\lim_{\rho\to\infty}\lim_{m\to\infty}Pr(ng(r_\rho^{-\epsilon})\eta(\varepsilon,\rho)\geq r_\rho^{-\delta}g(r_\rho^{-\epsilon})\eta(\varepsilon,\rho))$$

$$\leq \lim_{\rho\to\infty}\lim_{m\to\infty} r_\rho^\delta E(n)$$

where $\lim_{\rho\to\infty} r_\rho^{-\delta} g(r_\rho^{-\epsilon})\eta(\varepsilon,\rho) = 0$ due to (4), (41) and $\delta < 2\epsilon$, $\lim_{\rho\to\infty} r_\rho^B = 0$ for any positive constant B, and $\lim_{\rho\to\infty}\lim_{m\to\infty} r_\rho^\delta E(n) = 0$ due to that $\lim_{\rho\to\infty}\lim_{m\to\infty} E(n) = \lim_{\rho\to\infty} E(W^T) = e^{-b}$ is a bounded value and that $\lim_{\rho\to\infty} r_\rho^\delta = 0$. Therefore

$$\lim_{\rho\to\infty}\lim_{m\to\infty} Pr(ng(r_\rho^{-\epsilon})\eta(\varepsilon,\rho)=0) = 1 \tag{42}$$

As a result of (7), (40), (41) and (42):

$$\begin{aligned}
&\lim_{\rho\to\infty}\lim_{m\to\infty}\rho E(B_{U,i})\\
\leq\ &\lim_{\rho\to\infty}\lim_{m\to\infty}\rho E(e^{-\int_{D(\boldsymbol{x}_i,r_\rho^{1-\epsilon})}\rho g(\frac{\|\boldsymbol{x}-\boldsymbol{x}_i\|^T}{r_\rho})d\boldsymbol{x}})\\
=\ &\lim_{\rho\to\infty}\rho e^{-\int_{D(\mathbf{0},r_\rho^{1-\varepsilon})}\rho g(\frac{\|\boldsymbol{x}\|}{r_\rho})d\boldsymbol{x}}\\
=\ &\lim_{\rho\to\infty}\rho e^{-\rho r_\rho^2(C-\int_{\Re^2\setminus D(\mathbf{0},r_\rho^{-\varepsilon})} g(\|\boldsymbol{x}\|)d\boldsymbol{x})}\\
=\ & e^{-b}\lim_{\rho\to\infty} e^{\rho r_\rho^2\int_{\Re^2\setminus D(\mathbf{0},r_\rho^{-\varepsilon})} g(\|\boldsymbol{x}\|)d\boldsymbol{x}} = e^{-b}
\end{aligned} \tag{43}$$

where the last step results because

$$\begin{aligned}
&\lim_{\rho\to\infty}\rho r_\rho^2\int_{\Re^2\setminus D(\mathbf{0},r_\rho^{-\varepsilon})} g(\|\boldsymbol{x}\|)\,d\boldsymbol{x}\\
=&\lim_{\rho\to\infty}\frac{\int_{r_\rho^{-\varepsilon}}^\infty 2\pi x g(x)\,dx}{\frac{C}{\log\rho+b}}\\
=&\lim_{\rho\to\infty}\frac{\pi\varepsilon r_\rho^{-\varepsilon} g(r_\rho^{-\varepsilon}) r_\rho^{-\varepsilon-2}\frac{\log\rho+b-1}{C\rho^2}}{\frac{C}{\rho(\log\rho+b)^2}} \qquad (44)\\
=&\lim_{\rho\to\infty}\frac{\pi\varepsilon}{C}(\log\rho+b)^2 r_\rho^{-2\varepsilon} o_\rho\left(\frac{1}{r_\rho^{-2\varepsilon}\log^2(r_\rho^{-2\varepsilon})}\right) = 0 \qquad (45)
\end{aligned}$$

where L'Hï¿œpital's rule is used in reaching (44) and in the third step (5) is used. Using (20), (36) and (43), it can be shown that

$$\lim_{\rho\to\infty}\lim_{m\to\infty}\rho E(B_{U,i}) - m^2 p_i = 0 \tag{46}$$

As a result of (37), (39) and (46), $\lim_{\rho\to\infty}\lim_{m\to\infty} b_3 = 0$.

A combination of the analysis in subsections A, B and C completes this proof.

Appendix II: Proof of Theorem 6

For notational convenience, we prove the result for $\mathcal{G}\left(\mathcal{X}_\lambda, g, A_{\frac{1}{r_\rho}}\right)$ and the result is equally valid for $\mathcal{G}\left(\mathcal{X}_\rho, g_{r_\rho}, A\right)$. The proof is based on analyzing the number of components in $\mathcal{G}\left(\mathcal{X}_\lambda, g, A_{\frac{1}{r_\rho}}\right)$ of order greater than some integer M as $\rho\to\infty$. Specifically we will show that $\lim_{M\to\infty}\lim_{\rho\to\infty}\Pr(\xi_{>M}=1)=1$.

A direct analysis of $\Pr(\xi_{>M}=1)$ can be difficult. In this paper, we first analyze $E(\xi_{>M})$ and then use the result on $E(\xi_{>M})$ to establish the desired asymptotic result on $\Pr(\xi_{>M}=1)$.

Denote by $g_1(\boldsymbol{x}_1,\ldots,\boldsymbol{x}_k)$ the probability that a set of k nodes at non-random positions $\boldsymbol{x}_1,\ \ldots,\ \boldsymbol{x}_k \in A_{\frac{1}{r_\rho}}$ forms a connected component where nodes are connected randomly and independently following the connection function g. Denote by $g_2(\boldsymbol{y};\boldsymbol{x}_1,\boldsymbol{x}_2,\ldots,\boldsymbol{x}_k)$ the probability that a node at non-random position $\boldsymbol{y}$ is connected to at least one node in $\{\boldsymbol{x}_1,\boldsymbol{x}_2,\ldots,\boldsymbol{x}_k\}$. As an easy consequence of [10, Lemma 4], which showed that the expected number of components of order k, denoted by ξ_k , in $\mathcal{G}\left(\mathcal{X}_\lambda, g, A_{\frac{1}{r_\rho}}\right)$ is given by $E(\xi_k) = \frac{\lambda^k}{k!}\int_{(A_{\frac{1}{r_\rho}})^k} g_1(\boldsymbol{x}_1,\ldots,\boldsymbol{x}_k) e^{-\lambda\int_{A_{\frac{1}{r_\rho}}} g_2(\boldsymbol{y};\boldsymbol{x}_1,\ldots,\boldsymbol{x}_k)d\boldsymbol{y}} d(\boldsymbol{x}_1\cdots\boldsymbol{x}_k)$, it follows that

$$\begin{aligned}
&E(\xi_{>M})\\
=&\sum_{k=M+1}^\infty\frac{\lambda^k}{k!}\int_{(A_{\frac{1}{r_\rho}})^k}(g_1(\boldsymbol{x}_1,\ldots,\boldsymbol{x}_k)\\
&e^{-\lambda\int_{A_{\frac{1}{r_\rho}}} g_2(\boldsymbol{y};\boldsymbol{x}_1,\ldots,\boldsymbol{x}_k)d\boldsymbol{y}})d(\boldsymbol{x}_1\cdots\boldsymbol{x}_k)\\
\leq&\sum_{k=M+1}^\infty\frac{\lambda^k}{k!}\int_{(A_{\frac{1}{r_\rho}})^k} e^{-\lambda\int_{A_{\frac{1}{r_\rho}}} g_2(\boldsymbol{y};\boldsymbol{x}_1,\ldots,\boldsymbol{x}_k)d\boldsymbol{y}} d(\boldsymbol{x}_1\cdots\boldsymbol{x}_k)\\
=&\sum_{k=1}^\infty\frac{\lambda^k}{k!}\int_{(A_{\frac{1}{r_\rho}})^k} e^{-\lambda\int_{A_{\frac{1}{r_\rho}}} g_2(\boldsymbol{y};\boldsymbol{x}_1,\ldots,\boldsymbol{x}_k)d\boldsymbol{y}} d(\boldsymbol{x}_1\cdots\boldsymbol{x}_k)\\
&-\sum_{k=1}^M\frac{\lambda^k}{k!}\int_{(A_{\frac{1}{r_\rho}})^k} e^{-\lambda\int_{A_{\frac{1}{r_\rho}}} g_2(\boldsymbol{y};\boldsymbol{x}_1,\ldots,\boldsymbol{x}_k)d\boldsymbol{y}} d(\boldsymbol{x}_1\cdots\boldsymbol{x}_k)
\end{aligned} \tag{47}$$

In the following we show that as $\rho\to\infty$, the first term in (47) converges to $e^{e^{-b}}$, and the second term in (47) after the "$-$" sign is lower-bounded by $\sum_{k=1}^M\frac{(e^{-b})^k}{k!}$. The conclusion then follows that $E(\xi_{>M})$ converge to 1 as $\rho\to\infty$ and $M\to\infty$.

Let us consider the first term in (47) now. Let

$$\Phi \triangleq \lambda\int_{A_{\frac{1}{r_\rho}}}[1-g_2(\boldsymbol{y};\boldsymbol{x}_1,\ldots\boldsymbol{x}_k)]d\boldsymbol{y} \tag{48}$$

for convenience. It can be shown that

$$\begin{aligned}
&\lim_{\rho\to\infty}\sum_{k=1}^\infty\frac{\lambda^k}{k!}\int_{(A_{\frac{1}{r_\rho}})^k} e^{-\lambda\int_{A_{\frac{1}{r_\rho}}} g_2(\boldsymbol{y};\boldsymbol{x}_1,\ldots\boldsymbol{x}_k)d\boldsymbol{y}} d(\boldsymbol{x}_1,\ldots\boldsymbol{x}_k)\\
=&\lim_{\rho\to\infty}\sum_{k=1}^\infty\frac{\lambda^k}{k!}e^{-\rho}\int_{(A_{\frac{1}{r_\rho}})^k} e^{\Phi} d(\boldsymbol{x}_1,\ldots\boldsymbol{x}_k)\\
=&\lim_{\rho\to\infty}\sum_{k=1}^\infty\frac{\lambda^k}{k!}e^{-\rho}\int_{(A_{\frac{1}{r_\rho}})^k}\sum_{n=0}^\infty\frac{\Phi^n}{n!}d(\boldsymbol{x}_1,\ldots\boldsymbol{x}_k)\\
=&\sum_{n=0}^\infty\frac{1}{n!}\lim_{\rho\to\infty}\sum_{k=1}^\infty\frac{\lambda^k}{k!}e^{-\rho}\int_{(A_{\frac{1}{r_\rho}})^k}\Phi^n d(\boldsymbol{x}_1,\ldots\boldsymbol{x}_k)
\end{aligned} \tag{49}$$

Next we shall show that in (49), $\lim_{\rho\to\infty}\sum_{k=1}^\infty\frac{\lambda^k}{k!}e^{-\rho}\int_{(A_{\frac{1}{r_\rho}})^k}\Phi^n d(\boldsymbol{x}_1,\ldots\boldsymbol{x}_k) = (e^{-b})^n$. Given this result, conclusion readily follows from (49) that the first term in (47) converges to $e^{e^{-b}}$.

A direct computation of the term $\lim_{\rho\to\infty}\sum_{k=1}^\infty\frac{\lambda^k}{k!}e^{-\rho}\int_{(A_{\frac{1}{r_\rho}})^k}\Phi^n d(\boldsymbol{x}_1,\ldots\boldsymbol{x}_k)$ turns out

to be very difficult. To resolve the difficulty, we construct a random integer X, depending on ρ, such that on the one hand, the pmf (probability mass function) of X has an analytical form that can be easily related to the term $\sum_{k=1}^{\infty}\frac{\lambda^k}{k!}e^{-\rho}\int_{(A_{\frac{1}{r_\rho}})^k}\Phi^n d(\boldsymbol{x}_1,\ldots\boldsymbol{x}_k)$; and on the other hand using the Chen-Stein bound we are familiar with, the pmf can be shown to converge to a Poisson distribution as $\rho\to\infty$. In this way, we are able to compute the above term using the intermediate random integer X. In the following, we give details of the analysis.

We first construct the random integer X described in the last paragraph and demonstrate its properties related to our analysis.

Consider an additional *independent* Poisson point process $\mathcal{X}'_\lambda$ with nodes Poissonly distributed on $A_{\frac{1}{r_\rho}}$ and with density λ, being added to $\mathcal{G}\left(\mathcal{X}_\lambda, g, A_{\frac{1}{r_\rho}}\right)$. Further, nodes in $\mathcal{X}'_\lambda$ are connected with nodes in $\mathcal{X}_\lambda$ following g independently, i.e. a node in $\mathcal{X}'_\lambda$ and a node in $\mathcal{X}_\lambda$ separated by an Euclidean distance x are connected with probability $g(x)$, independent of any other connection.

Let X be the number of nodes in $\mathcal{X}'_\lambda$ that are *not* directly connected to any node in $\mathcal{X}_\lambda$. It is evident that, conditioned on $\mathcal{X}_\lambda=(\boldsymbol{x}_1,\ldots\boldsymbol{x}_k)$ where $\boldsymbol{x}_1,\ldots\boldsymbol{x}_k\in A_{\frac{1}{r_\rho}}$ and $|\mathcal{X}_\lambda|>0$, a randomly chosen node in $\mathcal{X}'_\lambda$ at location $\boldsymbol{y}$ is *not* directly connected to any node in $\mathcal{X}_\lambda$ with probability $1-g_2(\boldsymbol{y};\boldsymbol{x}_1,\ldots,\boldsymbol{x}_k)$, which is determined by its location only. It readily follows that the conditional distribution of X, i.e. $X|\mathcal{X}_\lambda=(\boldsymbol{x}_1,\ldots\boldsymbol{x}_k)$, is Poisson with mean $\lambda\int_{A_{\frac{1}{r_\rho}}}[1-g_2(\boldsymbol{y};\boldsymbol{x}_1,\ldots,\boldsymbol{x}_k)]d\boldsymbol{y}$ [9]. As a result of the above discussion:

$$\Pr(X=m|\mathcal{X}_\lambda=(\boldsymbol{x}_1,\ldots\boldsymbol{x}_k))=\frac{\Phi^m}{m!}e^{-\Phi} \tag{50}$$

Obviously when $\mathcal{X}_\lambda=\emptyset$, $\Pr(X=m|\mathcal{X}_\lambda=\emptyset)=\Pr(|\mathcal{X}'_\lambda|=m)$. Therefore the unconditional distribution of X is given by:

$$\begin{aligned}&\Pr(X=m)\\&=\sum_{k=1}^{\infty}\frac{\lambda^k}{k!}e^{-\rho}\int_{(A_{\frac{1}{r_\rho}})^k}\frac{\Phi^m}{m!}e^{-\Phi}d(\boldsymbol{x}_1,\ldots\boldsymbol{x}_k)+\frac{\rho^m}{m!}e^{-2\rho}\end{aligned} \tag{51}$$

Note that as $\rho\to\infty$, the term $\frac{\rho^m}{m!}e^{-2\rho}$ in (51), which is associated with $\mathcal{X}_\lambda=\emptyset$, becomes vanishingly small. Further note that $\sum_{m=0}^{\infty}\frac{\rho^m}{m!}e^{-2\rho}=e^{-\rho}\to 0$ as $\rho\to\infty$, i.e. as $\rho\to\infty$ even the cumulative contribution to the cdf of X is negligibly small.

If we define $g_2(\boldsymbol{y};\emptyset)\triangleq 0$ for completeness, we can also write (51) as

$$\Pr(X=m)=\sum_{k=0}^{\infty}\frac{\lambda^k}{k!}e^{-\rho}\int_{(A_{\frac{1}{r_\rho}})^k}\frac{\Phi^m}{m!}e^{-\Phi}d(\boldsymbol{x}_1,\ldots\boldsymbol{x}_k) \tag{52}$$

Using (52), it can be readily shown that

$$\begin{aligned}E(X)&=\sum_{m=0}^{\infty}m\Pr(X=m)\\&=\sum_{k=0}^{\infty}\frac{\lambda^k}{k!}e^{-\rho}\int_{(A_{\frac{1}{r_\rho}})^k}\Phi d(\boldsymbol{x}_1,\ldots\boldsymbol{x}_k)\\&=\sum_{k=0}^{\infty}\frac{\lambda^k}{k!}e^{-\rho}\{\lambda\int_{A_{\frac{1}{r_\rho}}}\{\int_{A_{\frac{1}{r_\rho}}}[1-g(\|\boldsymbol{x}-\boldsymbol{y}\|)]d\boldsymbol{x}\}^k d\boldsymbol{y}\}\\&=\lambda\int_{A_{\frac{1}{r_\rho}}}e^{-\lambda\int_{A_{\frac{1}{r_\rho}}}g(\|\boldsymbol{x}-\boldsymbol{y}\|)d\boldsymbol{x}}d\boldsymbol{y}\end{aligned} \tag{53}$$

Comparing the above equation with [10, Theorem 1], the conclusion readily follows that the above value is equal to the expected number of isolated nodes in $\mathcal{G}\left(\mathcal{X}_\lambda, g, A_{\frac{1}{r_\rho}}\right)$, denoted by W. It then follows from [10, Theorem 1], that $\lim_{\rho\to\infty}E(X)=e^{-b}$. In fact a stronger result that the distributions of X and W converge to the same Poisson distribution as $\rho\to\infty$ can be established:

Lemma 15. *As $\rho\to\infty$, the distribution of X converges to a Poisson distribution with mean e^{-b}, i.e. the total variation distance between the distribution of X and a Poisson distribution with mean e^{-b} reduces to 0 as $\rho\to\infty$.*

Lemma 15 can be proved using exactly the same steps as those used in proving Theorem 2. Therefore the proof is omitted.

As a result of Lemma 15, for an arbitrary set of non-negative integers, denoted by Γ,

$$\lim_{\rho\to\infty}\sum_{m\in\Gamma}\Pr(X=m)=\sum_{m\in\Gamma}\frac{(e^{-b})^m}{m!}e^{-e^{-b}} \tag{54}$$

Now we are ready to continue our analysis on $\lim_{\rho\to\infty}\sum_{k=1}^{\infty}\frac{\lambda^k}{k!}e^{-\rho}\int_{(A_{\frac{1}{r_\rho}})^k}\Phi^n d(\boldsymbol{x}_1,\ldots\boldsymbol{x}_k)$. Using (51) first and then using (54), it can be shown that for any positive integer n:

$$\begin{aligned}&\lim_{\rho\to\infty}\sum_{k=1}^{\infty}\frac{\lambda^k}{k!}e^{-\rho}\int_{(A_{\frac{1}{r_\rho}})^k}\Phi^n d(\boldsymbol{x}_1,\ldots\boldsymbol{x}_k)\\&=\lim_{\rho\to\infty}\sum_{k=1}^{\infty}\frac{\lambda^k}{k!}e^{-\rho}\int_{(A_{\frac{1}{r_\rho}})^k}\sum_{m=0}^{\infty}\Phi^n\frac{\Phi^m}{m!}e^{-\Phi}d(\boldsymbol{x}_1,\ldots\boldsymbol{x}_k)\\&=\sum_{m=0}^{\infty}\frac{1}{m!}\lim_{\rho\to\infty}\sum_{k=1}^{\infty}\frac{\lambda^k}{k!}e^{-\rho}\int_{(A_{\frac{1}{r_\rho}})^k}\Phi^{n+m}e^{-\Phi}d(\boldsymbol{x}_1,\ldots\boldsymbol{x}_k)\\&=\sum_{m=0}^{\infty}\frac{1}{m!}\lim_{\rho\to\infty}(\Pr(X=n+m)-\frac{\rho^{(n+m)}}{(n+m)!}e^{-2\rho})(n+m)!\\&=\sum_{m=0}^{\infty}\frac{(e^{-b})^{n+m}}{m!}e^{-e^{-b}}=(e^{-b})^n\end{aligned}$$

Using the above equation, it follows from (49) that

$$\begin{aligned}&\lim_{\rho\to\infty}\sum_{k=1}^{\infty}\frac{\lambda^k}{k!}\int_{(A_{\frac{1}{r_\rho}})^k}e^{-\lambda\int_{A_{\frac{1}{r_\rho}}}g_2(\boldsymbol{y};\boldsymbol{x}_1,\ldots\boldsymbol{x}_k)d\boldsymbol{y}}d(\boldsymbol{x}_1,\ldots\boldsymbol{x}_k)\\&=\sum_{n=0}^{\infty}\frac{(e^{-b})^n}{n!}=e^{e^{-b}}\end{aligned} \tag{55}$$

This deals with the first term on the right of (47). Now we continue with the analysis of the second term in (47). As an easy consequence of the union bound, $g_2\left(\boldsymbol{y};\boldsymbol{x}_1,\boldsymbol{x}_2,\ldots,\boldsymbol{x}_k\right) \leq \sum_{i=1}^{k} g\left(\|\boldsymbol{y}-\boldsymbol{x}_i\|\right)$, it can then be shown that

$$\begin{aligned}
&\frac{\lambda^k}{k!}\int_{(A_{\frac{1}{r_\rho}})^k} e^{-\lambda\int_{A_{\frac{1}{r_\rho}}} g_2(\boldsymbol{y};\boldsymbol{x}_1,\ldots\boldsymbol{x}_k)d\boldsymbol{y}} d(\boldsymbol{x}_1\cdots\boldsymbol{x}_k)\\
\geq\;&\frac{\lambda^k}{k!}\int_{(A_{\frac{1}{r_\rho}})^k} e^{-\lambda\int_{A_{\frac{1}{r_\rho}}} \sum_{i=1}^{k} g(\|\boldsymbol{y}-\boldsymbol{x}_i\|)d\boldsymbol{y}} d(\boldsymbol{x}_1\cdots\boldsymbol{x}_k)\\
=\;&\frac{\left(\lambda\int_{A_{\frac{1}{r_\rho}}} e^{-\lambda\int_{A_{\frac{1}{r_\rho}}} g(\|\boldsymbol{x}-\boldsymbol{y}\|)d\boldsymbol{y}} d\boldsymbol{x}\right)^k}{k!}
\end{aligned}\tag{56}$$

and using [10, Theorem 1], it can be further shown that

$$\begin{aligned}
&\lim_{\rho\to\infty}\frac{\lambda^k}{k!}\int_{(A_{\frac{1}{r_\rho}})^k} e^{-\lambda\int_{A_{\frac{1}{r_\rho}}} g_2(\boldsymbol{y};\boldsymbol{x}_1,\ldots\boldsymbol{x}_k)d\boldsymbol{y}} d(\boldsymbol{x}_1\cdots\boldsymbol{x}_k)\\
\geq&\frac{(e^{-b})^k}{k!}
\end{aligned}\tag{57}$$

Note that (57) can also be obtained from Jensen's inequality.

Combining (47), (55) and (57), it follows that

$$\lim_{\rho\to\infty} E(\xi_{>M}) \leq e^{e^{-b}} - \sum_{k=1}^{M}\frac{(e^{-b})^k}{k!} = 1 + \frac{(\eta_M)^{M+1}}{(M+1)!} \tag{58}$$

where in the last step Taylor's theorem is used, η_M is a number depending on M and $0 \leq \eta_M \leq e^{-b}$.

In Theorems 2 and [10, Theorem 4], we have established respectively that the asymptotic distribution of the number of isolated nodes in $\mathcal{G}\left(\mathcal{X}_\lambda, g, A_{\frac{1}{r_\rho}}\right)$ is Poisson with mean e^{-b} and the number of components in $\mathcal{G}\left(\mathcal{X}_\lambda, g, A_{\frac{1}{r_\rho}}\right)$ of order within $[2,M]$ vanishes as $\rho\to\infty$. As a consequence of the above two results,

$$\lim_{\rho\to\infty}\Pr(\xi_{>M}\geq 1) = 1 \text{ and } \lim_{\rho\to\infty}\Pr(\xi_{>M}=0)=0 \tag{59}$$

Further note that

$$\begin{aligned}
&E(\xi_{>M}) = \sum_{m=1}^{\infty} m\Pr(\xi_{>M}=m)\\
&\geq \Pr(\xi_{>M}=1) + 2\sum_{m=2}^{\infty}\Pr(\xi_{>M}=m)\\
&= \Pr(\xi_{>M}=1) + 2(1-\Pr(\xi_{>M}=1)-\Pr(\xi_{>M}=0))
\end{aligned}\tag{60}$$

Combing the three equations (58), (59) and (60):

$$\lim_{\rho\to\infty}\Pr(\xi_{>M}=1) \geq 1 - \frac{(\eta_M)^{M+1}}{(M+1)!} \tag{61}$$

As an easy consequence of the above equation:

$$\lim_{M\to\infty}\lim_{\rho\to\infty}\Pr\left(\xi_{>M}=1\right) = 1$$

References

[1] G. Mao and B. D. Anderson, "On the asymptotic connectivity of random networks under the random connection model," in *IEEE INFOCOM 2011*, pp. 1–9.

[2] P. Gupta and P. R. Kumar, *Critical Power for Asymptotic Connectivity in Wireless Networks*. Boston, MA: Birkhauser, 1998, pp. 547–566.

[3] M. D. Penrose, *Random Geometric Graphs*, ser. Oxford Studies in Probability. Oxford University Press, USA, 2003.

[4] M. Haenggi, J. G. Andrews, F. Baccelli, O. Dousse, and M. Franceschetti, "Stochastic geometry and random graphs for the analysis and design of wireless networks," *IEEE Journal on Selected Areas in Communications*, vol. 27, no. 7, pp. 1029–1046, 2009.

[5] M. D. Penrose, "On k-connectivity for a geometric random graph," *Random Structures and Algorithms*, vol. 15, no. 2, pp. 145–164, 1999.

[6] F. Xue and P. Kumar, "The number of neighbors needed for connectivity of wireless networks," *Wireless Networks*, vol. 10, no. 2, pp. 169–181, 2004.

[7] M. Penrose, "The longest edge of the random minimal spanning tree," *The Annals of Applied Probability*, vol. 7, no. 2, pp. 340–361, 1997.

[8] ——, "A strong law for the longest edge of the minimal spanning tree," *The Annals of Applied Probability*, vol. 27, no. 1, pp. 246–260, 1999.

[9] R. Meester and R. Roy, *Continuum Percolation*, ser. Cambridge Tracts in Mathematics. Cambridge University Press, 1996.

[10] G. Mao and B. D. Anderson, "Towards a better understanding of large scale network models," *IEEE/ACM Transactions on Networking*, vol. 20, no. 2, pp. 408 – 421, 2012, an extended version available at http://arxiv.org/abs/1012.5723.

[11] C. Bettstetter and C. Hartmann, "Connectivity of wireless multihop networks in a shadow fading environment," *Wireless Networks*, vol. 11, no. 5, pp. 571–579, 2005.

[12] D. Miorandi, "The impact of channel randomness on coverage and connectivity of ad hoc and sensor networks," *IEEE Transactions on Wireless Communications*, vol. 7, no. 3, pp. 1062–1072, 2008.

[13] C.-W. Yi, P.-J. Wan, X.-Y. Li, and O. Frieder, "Asymptotic distribution of the number of isolated nodes in wireless ad hoc networks with bernoulli nodes," *IEEE Transactions on Communications*, vol. 54, no. 3, pp. 510–517, 2006.

[14] M. Franceschetti and R. Meester, "Critical node lifetimes in random networks via the chen-stein method," *IEEE Transactions on Information Theory*, vol. 52, no. 6, pp. 2831–2837, 2006.

[15] ——, *Random Networks for Communication*. Cambridge University Press, 2007.

[16] R. Arratia, L. Goldstein, and L. Gordon, "Poisson approximation and the chen-stein method," *Statistical Science*, vol. 5, no. 4, pp. 403–434, 1990.

[17] A. D. Barbour, L. Holst, and S. Jason, *Poisson Approximation*. Oxford University Press, New York, 2003.

[18] G. Grimmett and D. Stirzaker, *Probability and Random Processes*, 3rd ed. Oxford, 2001.

[19] T. Yang, G. Mao, and W. Zhang, "Connectivity of large-scale csma networks," *IEEE Transactions on Wireless Communications*, vol. 11, no. 9, pp. 2266 – 2275, 2012.

[20] O. Dousse, F. Baccelli, and P. Thiran, "Impact of interferences on connectivity in ad hoc networks," *IEEE/ACM Transactions on Networking*, vol. 13, no. 2, pp. 425–436, 2005.

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